

An equivalent condition for a uniform space to be coverable

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Abstract

We prove that an equivalent condition for a uniform space to be coverable is that the images of the natural projections in the fundamental inverse system are uniformly open in a certain sense. As corollaries we (1) obtain a concrete way to find covering entourage, (2) correct an error in [3], and (3) show that coverable is equivalent to chain connected and uniformly joinable in the sense of [5]. Keywords: universal cover, uniform space, coverable, fundamental group MSC: 55Q52; 54E15, 55M10

In [6] and [3] we approached the problem of defining universal covers of locally bad spaces using the following ideas: (1) The appropriate category in which to work in is not topological spaces, but rather uniform spaces and uniformly continuous mappings. (2) The replacement for quotient mappings in this category is bi-uniformly continuous mappings (see below). (3) The appropriate replacement for curves is equivalence classes of chains. We showed in [3] that such a program can be carried out for a large class of uniform spaces called *coverable spaces*. In particular we constructed, for any uniform space X , a uniform space \tilde{X} , a natural uniformly continuous mapping $\phi : \tilde{X} \rightarrow X$ and a group $\delta_1(X)$ that acts on \tilde{X} . For coverable spaces the mapping ϕ has many of the properties of a universal covering map, such as lifting and universal properties, and we refer to the space \tilde{X} as the uniform universal cover of X . The group $\delta_1(X)$ (which we called the “deck group” in [3] but which was renamed the “uniform fundamental group” in [7]) is a functorial invariant of uniform structures having properties like the fundamental group in this category. See [3] for many specific theorems and examples, and [4] for some additional applications.

The space \tilde{X} is the inverse limit of the fundamental inverse system (X_E, ϕ_{EF}) of X , which is indexed on the set of all entourages E of X . Roughly speaking, X_E “unrolls” nontrivial classes of loops that are in some sense larger than E

(we will give more background below). We will denote by $\phi_E : \tilde{X} \rightarrow X_E$ and $\phi : \tilde{X} \rightarrow X$ the natural projections (the latter is actually just the endpoint mapping). X is coverable by definition if ϕ is surjective ϕ_E is surjective for all E in some basis (called a covering basis) for the uniform structure of X . Elements of the covering basis are called covering entourages. Existence of a covering basis can be proved in many cases—for example it is not too difficult to show that connected and uniformly locally connected pseudometric spaces—which includes all geodesic spaces—are coverable (Theorem 98, [3]). On the other hand, it is easy to find in coverable spaces entourages that are not covering entourages (cf. Example 16), and without uniform nice local conditions it can be difficult to verify coverability. In this paper we show that coverability is equivalent to the following: X is chain connected and for any entourage E , ϕ_E has image that is *uniformly open* in X_E in a sense defined below (Theorem 12). While surjectivity of maps in an inverse system is generally a strong condition useful for proving theorems, from the standpoint of verification there is a clear advantage in not having to hunt for the covering entourages. Theorem 12 provides a condition that may be verified for *arbitrary* entourages. Moreover, as a corollary we obtain a constructive method for extracting a covering entourage from an arbitrary entourage without having to consider the mapping ϕ_E at all (see Corollary 15 and Example 16).

In [5] the authors explore our construction of the uniform universal cover in the setting of what they call *uniformly joinable* uniform spaces. At first their construction does not look like our construction, and one must look in Section 8 of [5] to find a statement that they are “identical”. As we explain below, their definitions of $GP(X, *)$ and $\check{\pi}(X, *)$ are simply translations of our definitions of \tilde{X} and $\delta_1(X)$ into the language of Rips complexes. Moreover, another application of our main theorem is that the class of chain connected, uniformly joinable spaces considered in [5] is precisely the same as the class of coverable spaces (Corollary 17). The authors of [5] state that “a topologist would be skeptical” of this particular result—an assertion backed by some musings on Siebenmann’s thesis. Nonetheless, their definition is closely related to concepts in continua theory and they obtain the interesting result that a metric compactum X is uniformly joinable if and only if the function $\phi : \tilde{X} \rightarrow X$ is surjective. In light of Corollary 17, for metrizable spaces this is a strong generalization of the fact, proved in [2], that a compact topological group is coverable if and only if ϕ is surjective. Also in [5] the authors introduce a notion of generalized cover in the uniform category that does not require a group action. In light of this, what we called “covers of uniform spaces” in [3] really should be called something like “regular uniform covers” as is suggested in [5].

We do not use any theorems from [5] in our proofs and in fact, in light of Corollary 17, some of the theorems in [5] were already proved in [3]. Thanks to Jurek Dydak for critiques and stimulating comments. In particular, he pointed out an error in [3] that is corrected in the present paper. Valera Berestovskii provided some valuable comments.

We will use the notation of [3]. In particular, we generally use f in place of

$f \times f$; for example, if E is an entourage in a uniform space we will write $f(E)$ rather than $(f \times f)(E)$. For an entourage E we let $B(x, E) := \{y : (x, y) \in E\}$. Given a uniform space X , for any entourage E , X_E is defined to be the space of E -homotopy classes of E -chains $\alpha := \{x_0 = *, \dots, x_n\}$, where $*$ is a basepoint. By definition, α is an E -chain if $(x_i, x_{i+1}) \in E$ for all i . An E -homotopy of α is a finite sequence of moves starting with α , where each move consists of adding or taking away a point (but not endpoints!) so long as doing so results in an E -chain. For chain connected spaces (meaning every pair of points is joined by an E -chain for all E) nothing of consequence depends on the choice of basepoint so we generally eliminate it from the notation. The space X_E , the elements of which are denoted $[\alpha]_E$, has a natural uniform structure having a basis consisting of sets F^* , where $F \subset E$ and $([\alpha]_E, [\beta]_E) \in F^*$ if and only if $[\alpha]_E = [x_0, \dots, x_{n-1}, x]_E$, and $[\beta]_E = [x_0, \dots, x_{n-1}, y]_E$, with $(x, y) \in F$. \tilde{X} is given the inverse limit uniformity. When $F \subset E$, the mapping $\phi_{EF} : X_F \rightarrow X_E$ simply considers an F -chain as an E -chain, i.e., $\phi_{EF}([\alpha]_F) = [\alpha]_E$, and $\phi_{XE} : X_E \rightarrow X$ is the endpoint mapping. With respect to the natural uniform structure these mappings are bi-uniformly continuous in the sense that the inverse image of any entourage is an entourage, and the image of any entourage is an entourage in the subspace uniformity of the image of the mapping. Given a uniformly continuous mapping $f : X \rightarrow Y$ and entourages E, F in X, Y , respectively, such that $f(E) \subset F$, there is a unique basepoint-preserving induced uniformly continuous function $f_{EF} : X_E \rightarrow Y_F$ such that $\phi_{YF} \circ f_{EF} = f \circ \phi_{XE}$, which simply takes $[\alpha]_E$ to $[f(\alpha)]_F$.

If X is chain connected, the function $\phi_{XE} : X_E \rightarrow X$ is a quotient mapping via the isomorphic action of the group $\delta_E(X)$ consisting of E -homotopy classes of E -loops. Precisely what this means is not needed for this paper (see [6] for the definitions); we can get by with two facts: first, if $\phi_{XE}(a) = \phi_{XE}(b)$ then for some $g \in \delta_E(X)$, $g(a) = b$ and second, the entourages F^* are *invariant* in the sense that for every $g \in \delta_E(X)$, $g(F^*) = F^*$.

Definition 1 *We say that a subset A of a uniform space X is uniformly open if there is an entourage E in X such that for every $a \in A$, $B(a, E) \subset A$.*

There are a few obvious facts: if A is a uniformly open set then A is open, the complement of A is uniformly open, and hence A itself is also closed. But for example in the rational numbers \mathbb{Q} with the usual metric there are plenty of open and closed subsets that are not uniformly open. The inverse image of any uniformly open set via a uniformly continuous function is uniformly open, but in general nothing can be said of images. For example consider the bi-uniformly continuous surjection $f : [0, 2] \times \mathbb{Z}_2 \rightarrow [0, 2]$ defined by $(q, 0) \mapsto q$ and $(q, 1) \mapsto \frac{q}{2}$. Here $[0, 2]$ has its usual metric, \mathbb{Z}_2 has the discrete metric, and $[0, 2] \times \mathbb{Z}_2 \rightarrow [0, 2]$ has the product metric. It is easy to check that $[0, 2] \times \{1\}$ is uniformly open in $[0, 2] \times \mathbb{Z}_2$ but of course $f([0, 2] \times \{1\}) = [0, 1]$ is not even open in $[0, 2]$. (But see Remark 6 below.)

Lemma 2 *A uniform space X is chain connected if and only if the only non-empty uniformly open subset of X is X .*

Proof. Suppose X is chain connected and let U be a nonempty uniformly open subset of X . If E is an entourage as in the definition of uniformly open, then any E -chain starting at x cannot leave U and so $U = X$. For the converse, let $x \in X$, E be an entourage, and U be the set of all points that are joined to x by an E -chain. If $z \in U$ then clearly $B(z, E) \in U$; hence U is uniformly open and non-empty, hence equal to X . Since E and x were arbitrary, X is chain connected. ■

Obviously the intersection of any two uniformly open subsets is uniformly open. As a corollary of this and the above lemma we obtain:

Corollary 3 *If any two uniformly open chain connected subsets of a uniform space X have non-empty intersection then they must be equal.*

Definition 4 *If X is a uniform space, $E \subset F$ are entourages in X , and A is a uniformly open subset of X_E , define*

$$F_A := \phi_{XE}(F^* \cap (A \times A)).$$

Lemma 5 *Let X be a chain connected uniform space, E be an entourage in X , and A be a uniformly open subset of X_E . Then $\phi_{XE}(A) = X$ and for any entourage $F \subset E$, F_A is an entourage in X .*

Proof. Consider entourages $W \subset F \subset E$ such that if $x \in A$ and $(x, y) \in W^*$ then $y \in A$. We will first prove that $\phi_{XE}(A)$ is uniformly open and hence equal to X . Suppose that $(a, b) \in W$ and $a \in \phi_{XE}(A)$. So there exist $z \in A$ such that $\phi_{XE}(z) = a$ and since $W = \phi_{XE}(W^*)$ (cf. [3], Proposition 16) there exists $(x, y) \in W^*$ such that $\phi_{XE}(x, y) = (a, b)$. Next there exists some $g \in \delta_E(X)$ such that $g(x) = z$. By the invariance of W^* , if $w := g(y)$ then $(z, w) \in W^*$ and $\phi_{XE}(z, w) = (a, b)$. By choice of W^* , $w \in A$, which places $b \in \phi_{XE}(A)$, finishing the proof that $\phi_{XE}(A)$ is uniformly open and equal to X . Now the initial assumption that $a \in \phi_{XE}(A)$ is superfluous and the same argument shows $W \subset W_A$ and since $W_A \subset F_A$, F_A is an entourage. ■

Remark 6 *The same proof as in the previous lemma shows the following: If $f : X \rightarrow Y$ is a quotient of uniform spaces via an isomorphic action and $A \subset X$ is uniformly open then $f(A)$ is uniformly open in Y (see [6] for a discussion of isomorphic actions).*

Corollary 7 *If X is a chain connected uniform space and there is some entourage E such that $\phi_E(\tilde{X})$ is uniformly open in X_E then $\phi : \tilde{X} \rightarrow X$ is surjective.*

Lemma 8 *Let X be a chain connected uniform space and E be an entourage in X such that $A := \phi_E(\tilde{X})$ is uniformly open. If $\alpha := \{ * = x_0, \dots, x_n \}$ is an E_A -chain then $[\alpha]_E \in A$.*

Proof. We will show by induction that $[x_0, \dots, x_k]_E \in A$ for all $k \leq n$. Certainly the statement is true for $k = 0$. Suppose that $[x_0, \dots, x_k]_E \in A$. Now

$(x_k, x_{k+1}) \in E_A$ and by definition there exist E -chains $\gamma := \{ * = y_0, \dots, y_m, x_k \}$ and $\omega := \{ y_0, \dots, y_m, x_{k+1} \}$ such that $[\gamma]_E, [\omega]_E \in A$ and $(x_k, x_{k+1}) \in E$. So we have $([a_D]_D), ([b_D]_D), ([c_D]_D) \in \tilde{X}$ such that $[a_E]_E = [x_0, \dots, x_k]_E$, $[b_E]_E = [\gamma]_E$, and $[c_E]_E = [\omega]_E$. Consider $\kappa := ([a_D * b_D^{-1} * c_D]_D) \in \tilde{X}$, where “ $*$ ” denotes concatenation of chains. Now

$$\begin{aligned} \phi_E(\kappa) &= [a_E * b_E^{-1} * c_E]_E = [\{x_0, \dots, x_k\} * \gamma^{-1} * \omega]_E \\ &= [x_0, \dots, x_k, y_m, \dots, y_1, y_0, y_1, \dots, y_m, x_{k+1}]_E = [x_0, \dots, x_{k+1}]_E \end{aligned}$$

where the last E -homotopy successively removes $y_0, y_1, y_1, y_2, \dots, y_m$. ■

Proposition 9 *Let X be a chain connected uniform space and E be an entourage in X such that $A := \phi_E(\tilde{X})$ is uniformly open. Letting $D := E^* \cap (A \times A)$ and $G := \phi_E^{-1}(D) = \phi_E^{-1}(E^*)$, there is a uniformly continuous function $\psi : \tilde{X} \rightarrow A_D$ such that the following diagram commutes*

$$\begin{array}{ccc} \tilde{X}_G & \xrightarrow{\phi_{\tilde{X}G}} & \tilde{X} \\ \downarrow \theta & \swarrow \psi & \downarrow \phi_E \\ A_D & \xrightarrow{\phi_{AD}} & A \end{array} \quad (1)$$

where $\theta = (\phi_E)_{GD}$ is the mapping induced by ϕ_E .

Proof. First of all we recall what it means for $([\alpha]_E, [\beta]_E) \in D$: $[\alpha]_E, [\beta]_E \in A$ and

$$[\alpha]_E = [y_0, \dots, y_m, x]_E \text{ and } [\beta]_E = [y_0, \dots, y_m, y]_E$$

for some choice of y_0, \dots, y_m , with $(x, y) \in E$. We will define $\psi := f \circ \phi_{EA}$, where

$$f([x_0, \dots, x_n]_{EA}) = [[x_0]_E, [x_0, x_1]_E, \dots, [x_0, \dots, x_n]_E]_D \quad (2)$$

We need to check various things about f . First of all, note that by Lemma 8, $[x_0, \dots, x_k]_E \in A$ for all $1 \leq k \leq n$ and therefore

$$\begin{aligned} &([x_0, \dots, x_k]_E, [x_0, \dots, x_k, x_{k+1}]_E) \\ &= ([x_0, \dots, x_k, x_k]_E, [x_0, \dots, x_k, x_{k+1}]_E) \in D \end{aligned}$$

so the definition at least goes into the correct set. To see that f is well-defined, consider an E_A -chain $\alpha' := \{x_0, \dots, x_{k-1}, x, x_k, \dots, x_n\}$, which would lead to

$$\begin{aligned} &[[x_0]_E, \dots, [x_0, \dots, x_{k-1}]_E, [x_0, \dots, x_{k-1}, x]_E, \\ &[x_0, \dots, x_{k-1}, x, x_k]_E, \dots, [x_0, \dots, x_{k-1}, x, x_k, \dots, x_n]_E]_D \end{aligned}$$

in the above definition. But notice that for any $k \leq m \leq n$, we already know $\{x_0, \dots, x_{k-1}, x, x_k, \dots, x_m\}$ and $\{x_0, \dots, x_{k-1}, x_k, \dots, x_m\}$ are E_A -homotopic, hence E -homotopic, so we can simply remove the “ x ” from all such terms. This leaves the one extra term $[x_0, \dots, x_{k-1}, x]_E$. But since $([x_0, \dots, x_{k-1}]_E, [x_0, \dots, x_{k-1}, x]_E) \in D$, up to D -homotopy we may simply remove this term, getting us back to (2).

We will now check that f is uniformly continuous. To do so we will have to be a little more careful with notation. Given an entourage $W \subset E_A$ in X we have an entourage called W^* in X_E and one called $W^\#$ in X_{E_A} . We will refer to the latter as $W^\#$. We also have the entourage $(W^* \cap (A \times A))^*$ in A_D , which we will simply denote by W^{**} . The proof of uniform continuity will be finished if we can show that $f(W^\#) \subset W^{**}$. Let $([\alpha]_{E_A}, [\beta]_{E_A}) \in W^\#$. By definition we may take $\alpha = \{x_0, \dots, x_n, x\}$ and $\beta = \{x_0, \dots, x_n, y\}$ with $(x, y) \in W$. We have

$$f([\alpha]_{E_A}) = [[x_0]_E, \dots, [x_0, \dots, x_n]_E, [x_0, \dots, x_n, x]_E]_D$$

$$f([\beta]_{E_A}) = [[x_0]_E, \dots, [x_0, \dots, x_n]_E, [x_0, \dots, x_n, y]_E]_D$$

Since $(x, y) \in W$ and $[x_0, \dots, x_n, x]_E, [x_0, \dots, x_n, y]_E \in A$ (Lemma 8 again), $([x_0, \dots, x_n, x]_E, [x_0, \dots, x_n, y]_E) \in W^* \cap (A \times A)$ and $(f([\alpha]_{E_A}), f([\beta]_{E_A})) \in W^{**}$.

We will now check the commutativity of the diagram. Suppose that $\eta := \{y_0 = *, y_1, \dots, y_n\}$ is a G -chain in \tilde{X} . This means that for all i , $(\phi_E(y_i), \phi_E(y_{i+1})) \in D$. In particular, $(\phi_E(y_0), \phi_E(y_1)) = ([*]_E, \phi_E(y_1)) \in D$. This means that we may write $\phi_E(y_1) = [* = w_0, \dots, w_m, x_1]_E$, where x_1 is the endpoint of y_1 , $\{w_0, \dots, w_m, *\}$ is E -homotopic to the identity and $(*, x_1) \in E$. But then we may use the null E -homotopy of $\{w_0, \dots, w_m, *\}$ to see that

$$\phi_E(y_1) = [* = w_m, *, x_1]_E = [* = x_1]_E.$$

By definition of D we also have $[*, x_1]_E = \phi_E(y_1) \in A$, which implies that $\{*, x_1\}$ is an E_A -chain. Proceeding inductively with essentially the same argument, we see that $\phi_E(y_i) = [* = x_1, \dots, x_i]_E$, where x_i is the endpoint of $\phi_E(y_i)$ and $\{x_0, \dots, x_n\}$ is an E_A -chain. By definition of θ ,

$$\begin{aligned} \theta([\eta]_G) &= [\phi_E(y_0), \dots, \phi_E(y_n)]_D = [[*]_E, [* = x_1]_E, \dots, [* = x_1, \dots, x_n]_E]_D \\ &= \psi(y_n) = \psi \circ \phi_{\tilde{X}G}([\eta]_G) \end{aligned}$$

This proves the commutativity of the upper triangle. The commutativity of the lower triangle is obvious from the definition of ψ . ■

Universal uniform spaces and universal bases were defined in [3]; the definitions will be explained in the proof below.

Proposition 10 *If X is a chain connected uniform space such that for every entourage E , $\phi_E : \tilde{X} \rightarrow X_E$ has uniformly open image in X_E then \tilde{X} is universal with an invariant (with respect to the action of $\delta_1(X)$) universal basis.*

Proof. Consider the diagram (1). We will start by showing that \tilde{X} is chain connected. Since ϕ_E is surjective onto A , so is ϕ_{AD} . This means that every pair of points in A is joined by a D -chain. Equivalently, $A \times A = \bigcup_{n=1}^{\infty} D^n$ (where D^n is the set of all points in A joined to the basepoint by a D -chain of length n). Since ϕ_E is surjective, it is easy to check that $\phi_E^{-1}(D^n) = (\phi_E^{-1}(D))^n = (\phi_E^{-1}(E^*))^n$ and so $\tilde{X} = \bigcup_{n=1}^{\infty} (\phi_E^{-1}(E^*))^n$ (cf. the proof of Lemma 11 in [3]). This means

that every pair of points in \tilde{X} is joined by a $\phi_E^{-1}(E^*)$ -chain. Since the set of all $\phi_E^{-1}(E^*)$ forms a basis for the uniformity of \tilde{X} , \tilde{X} is chain connected. This now implies that the mapping $\phi_{\tilde{X}G}$ is surjective and the hypotheses of Proposition 33 in [3] are satisfied for this diagram, implying that $\phi_{\tilde{X}G}$ is a uniform homeomorphism. By definition G is a universal entourage, and since it is of the form $\phi_E^{-1}(E^*)$, it is invariant (cf. Proposition 41 of [6]). We have shown that \tilde{X} has a basis of universal entourages; by definition this makes \tilde{X} universal. ■

Alas, the proof of Corollary 61 in [3] is not correct—or rather, the proof is correct for a weaker statement. The penultimate sentence in the proof requires an additional assumption. For example, the proof is correct for the following statement:

Lemma 11 *If $f : X \rightarrow Y$ is a quotient via an action on a uniform space X and X has a universal basis that is invariant with respect to the action, then Y is coverable.*

Corollary 61 was only used to establish the equivalence of the definition of “coverable topological group” as defined in [1] with the definition in [3] when applied to topological groups considered as uniform spaces. In particular, none of the results of [1] cited in the current paper relies on this corollary; Lemma 11 will suffice to prove our main Theorem 12, of which Corollary 61 is a corollary.

Theorem 12 *For a chain connected uniform space X , the following are equivalent:*

1. X is coverable.
2. $\phi : \tilde{X} \rightarrow X$ is a bi-uniformly continuous surjection.
3. For each entourage E in X and any choice of basepoint, $\phi_E(\tilde{X})$ is uniformly open in X_E .

Proof. $1 \Rightarrow 2$ follows from Theorem 45 in [3]. For $2 \Rightarrow 3$, let E be an entourage in X , $A := \phi_E(\tilde{X})$. Suppose that $([\alpha]_E, [\beta]_E) \in E_A^*$ and $[\alpha]_E \in A$. So there is some $([\gamma_D]_D) \in \tilde{X}$ such that $[\gamma_E]_E = [\alpha]_E$ and we may write $\alpha := \{x_0, \dots, x_n, x\}$ and $\beta := \{x_0, \dots, x_n, y\}$, with $(x, y) \in E_A$. This in turn means that we have $([\alpha_D]_D), ([\beta_D]_D) \in \tilde{X}$ with endpoints x and y such that $([\alpha_E]_E), ([\beta_E]_E) \in E^* \cap (A \times A)$. So we may now write $\alpha_E := \{y_0, \dots, y_m, x\}$ and $\beta_E := \{y_0, \dots, y_m, y\}$ and $(x, y) \in E$. Consider $([\gamma_D * \alpha_D^{-1} * \beta_D]_D) \in \tilde{X}$. Using an E -homotopy like the one in the proof of Lemma 8 we have

$$\begin{aligned} [\gamma_E * \alpha_E^{-1} * \beta_E]_E &= [x_0, \dots, x_n, x, y_m, \dots, y_0, y_1, \dots, y_m, y]_E \\ &= [x_0, \dots, x_n, x, y]_E = [\beta]_E \end{aligned}$$

This implies that $[\beta]_E \in A$ and finishes the proof that A is uniformly open.

To prove $3 \Rightarrow 1$, note that by Proposition 10, \tilde{X} has an invariant universal basis with respect to the isomorphic action of $\delta_1(X)$. Corollary 7 and Lemma 5 together show that ϕ is a bi-uniformly continuous surjection, hence a quotient with respect to this action (cf. Theorem 11, [6]). Lemma 11 now finishes the proof. ■

The next corollary is the statement Corollary 61 in [3]:

Corollary 13 *If $f : X \rightarrow Y$ is a bi-uniformly continuous surjection where X is universal and Y is uniform then Y is coverable.*

Proof. According to Proposition 57 in [3] we have the lift $f_L : X \rightarrow \tilde{Y}$ which satisfies $\phi \circ f_L = f$, where $\phi : \tilde{Y} \rightarrow Y$ is the projection. But then ϕ must be a uniformly continuous surjection. If E is an entourage in \tilde{Y} , then since f is bi-uniformly continuous, $f(f_L^{-1}(E))$ is an entourage that is contained in $\phi(E)$. This proves that ϕ is bi-uniformly continuous and hence Y is coverable by Theorem 12. ■

Corollary 14 *If X is coverable then E is a covering entourage if and only if X_E is chain connected.*

Proof. If E is a covering entourage then by definition $\phi_E : \tilde{X} \rightarrow X_E$ is surjective. Since \tilde{X} is chain connected, so is X_E . Conversely, if X_E is chain connected then by Lemma 2 and the third part of Theorem 12, ϕ_E must be surjective. ■

Note that the argument $3 \Rightarrow 1$ in the proof of Theorem 12 is constructive; it actually provides a covering basis. We can now sort through the steps to help identify this basis. The proof of Lemma 11, which is actually in [3], shows that the covering entourages are of the form $\phi(G)$, where G is an invariant universal entourage in \tilde{X} . The universal entourages in \tilde{X} come from Proposition 10, and they are of the form $\phi_E^{-1}(E^*)$ for any E . Letting $A := \phi_E(\tilde{X})$ we have

$$\phi(\phi_E^{-1}(E^*)) = \phi(\phi_E^{-1}(E^* \cap (A \times A))) = \phi_{X_E} \circ \phi_E \circ \phi_E^{-1}(E^* \cap (A \times A)) = E_A$$

Note that E_A is chain connected since \tilde{X} is. Combining this with Corollary 3 we obtain:

Corollary 15 *Let X be a coverable uniform space. For any entourage E , X_E has a unique chain connected uniformly open set A containing the basepoint, and E_A is a covering entourage.*

Example 16 *We will illustrate how Corollary 15 extracts a covering entourage from a non-covering entourage in the topological group \mathbb{R} . In a topological group with left uniformity, entourages are completely determined by symmetric open subsets of the identity (which always serves as the basepoint). For example, in \mathbb{R} , if U is any such set, there corresponds an entourage $E(U) := \{(x, y) : x - y \in U\}$. The open set corresponding to $E(U)^*$ in $\mathbb{R}_U := \mathbb{R}_{E(U)}$ is denoted by U^* . Using open sets rather than entourages makes it easier to see what is going on. In*

Example 48 of [1] we considered the set $U := (-1, 1) \cup (2, 4) \cup (-4, -2)$. In this example the components of U are far enough apart that \mathbb{R}_U consists of the topological group $\mathbb{R} \times \mathbb{Z}$ with the product uniform structure (\mathbb{Z} is discrete). The idea here is that the two outer components of U cannot be reached from 0 by $(-1, 1)$ -chains, so for example the equivalence class of the chain $\{0, 3\}$ lies in a different component from the identity component $A := \mathbb{R} \times \{0\}$ in \mathbb{R}_U . Along these lines, it is not hard to show that

$$U^* = (-1, 1) \times \{0\} \cup (2, 4) \times \{1\} \cup (-4, -2) \times \{-1\}$$

That is, the two outer components of U^* do not lie in A , which clearly is the unique chain connected uniformly open set containing the identity in $\mathbb{R} \times \mathbb{Z}$. Now we have $\phi_{\mathbb{R}_E(U)}(U^* \cap A) = (-1, 1) := V$. Since V is connected and \mathbb{R} is simply connected, $E(V)$ is a covering entourage (cf. [1]).

In [5] the notion of Rips complex is extended from metric spaces to uniform spaces: $R(X, E)$ is the subcomplex of the full complex over X having as simplices all $\{x_0, \dots, x_n\}$ such that $(x_i, x_j) \in E$ for all i and j . According to ([8], Section 3.6), any path in $R(X, E)$ is, up to homotopy, uniquely identified with a simplicial path, which in turn is uniquely determined by its vertices. These vertices, obviously, form an E -chain, and the basic moves in a fixed-endpoint simplicial homotopy of simplicial paths (adding or removing a pair of edges that span 2-simplex with one edge already in the path) correspond precisely to the basic moves in an E -homotopy (adding or removing a point so as to preserve that one has an E -chain). That is, the set of all fixed-endpoint homotopy equivalence classes of paths in $R(X, E)$ starting at a base point $*$ is naturally identified with X_E .

Using this natural identification of fixed-endpoint homotopies of paths in $R(X, E)$ and E -homotopies of E -chains, we will translate the basic definitions of [5]. Two E -chains $\alpha := \{x_0, \dots, x_n\}$ and $\beta := \{y_0, \dots, y_k\}$ (maybe without the same endpoints) are said in [5] to be E -homotopic if (1) $(x_0, y_0), (x_n, y_k) \in E$ and (2) β is (fixed-endpoint) E -homotopic to $\{y_0, x_0, \dots, x_n, y_k\}$. If α and β have the same pair of endpoints then of course “ E -homotopic” has the same meaning as in [3]. If α and β only have the same *starting* point $x_0 = y_0 = *$, then it is easy to check that α and β are E -homotopic precisely when $([\alpha]_E, [\beta]_E) \in E^*$. A *generalized curve from x to y* is defined in [5] to be a collection $\{[c_E]_E\}$ of E -homotopy classes of E -chains joining x and y such that if $F \subset E$ then $[c_F]_E = [c_E]_E$. The set of all generalized curves starting at $*$ is called $GP(X, *)$ in [5], but this set is obviously none other than \tilde{X} via the identification $\{[c_E]_E\} \leftrightarrow ([c_E]_E)$. The authors define a “natural uniform structure” on $GP(X, *)$ (a.k.a. \tilde{X}) by taking, for each entourage F in X , the set of all pairs $(\{[c_E]_E\}, \{[d_E]_E\})$ such that c_F is F -homotopic to d_F . Since c_F and d_F both start at $*$, as pointed out above this is equivalent to $([c_F]_F, [d_F]_F) \in F^*$. That is, the basis that they define consists precisely of the sets $\phi_F^{-1}(F^*)$, which of course is a basis for the inverse limit uniform structure on \tilde{X} . In other words, $GP(X, *)$ and \tilde{X} are one and the same space. Moreover, the mapping $\pi_X : GP(X, *) \rightarrow X$ of [5] is

the endpoint mapping (identical to $\phi : \tilde{X} \rightarrow X$), and the uniform fundamental group $\check{\pi}(X, *)$ of [5] is $\pi_X^{-1}(*) = \phi^{-1}(*)$ (“generalized loops”) with operation induced by concatenation (identical to $\delta_1(X)$).

According to [5], a uniform space X is called *joinable* if every pair of points in X is joined by a generalized curve; clearly this is equivalent to the surjectivity of $\phi : \tilde{X} \rightarrow X$. In [5] X is called *uniformly joinable* if for every entourage E there is an entourage F such that whenever $(x, y) \in F$, x and y are joined by a generalized curve $\{[c_D]_D\}$ that is “ E -short” in the sense that $[c_E]_E = [\{x, y\}]_E$.

Corollary 17 *If X is a chain connected uniform space then X is coverable if and only if X is uniformly joinable.*

Proof. Suppose that $\phi_E(\tilde{X})$ is uniformly open for all E ; so there is some $F \subset E$ such that if $([\alpha]_E, [\beta]_E) \in F^*$ and $[\alpha]_E \in \phi_E(\tilde{X})$ then $[\beta]_E \in \phi_E(\tilde{X})$. Let $(x, y) \in F$. Since X is chain connected there is some F -chain $\alpha = \{x_0 = *, \dots, x_{n-1}, x\}$ and we may let $\beta := \{x_0, \dots, x_{n-1}, x, y\}$. Note that since

$$([x_0, \dots, x_i]_E, [x_0, \dots, x_i, x_{i+1}]_E) \in F^*$$

for all i , it follows by induction on i that $[\alpha]_E \in \phi_E(\tilde{X})$. Likewise $[\beta]_E \in \phi_E(\tilde{X})$ and we have $[\alpha]_E = [\alpha_E]_E$ for some $([\alpha_D]_D) \in \tilde{X}$ and $[\beta]_E = [\beta_E]_E$ for some $([\beta_D]_D) \in \tilde{X}$. But the concatenated generalized curve $\{[\alpha_D^{-1} * \beta_D]_D\}$ certainly satisfies the E -short condition $[\alpha_E^{-1} * \beta_E]_E = [\{x, y\}]_E$; in fact one may remove the points $x_0, x_1, x_1, \dots, x_{n-1}, x_{n-1}, x$ in succession to create an E -homotopy between $\alpha_E^{-1} * \beta_E$ and $\{x, y\}$.

If X is uniformly joinable and E is an entourage, by definition there is some entourage $F \subset E$ such that if $(x, y) \in F$, x and y are joined by an E -short generalized curve. Let $\alpha = \{x_0 = *, \dots, x_n\}$ be an E -chain with $[\alpha]_E \in \phi_E(\tilde{X})$. If $([\alpha]_E, [\beta]_E) \in F^*$ then by definition of F^* we may assume that β is of the form $\{x_0 = *, \dots, x_{n-1}, x\}$ with $(x, x_n) \in F$. That is, there is an E -short generalized curve $\{[c_D]_D\}$ joining x_n and x with $c_E = \{x_n, x\}$. Now if $\phi_E([\alpha_D]_D) = [\alpha]_E$ then $g := ([\alpha_D * c_D]_D) \in \tilde{X}$ satisfies $\phi_E(g) = [\beta]_E$. ■

Remark 18 *In light of Corollaries 7 and 17 we have a very nice way to distinguish between joinable and uniformly joinable for a chain connected uniform space X , namely that for a joinable space, ϕ_E has uniformly open image for some E , while for a uniformly joinable space, ϕ_E has uniformly open image for all E .*

Remark 19 *Note that the equivalence of Theorem 12.2 and uniform joinability for a chain connected space was proved in [5] using completely different arguments.*

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